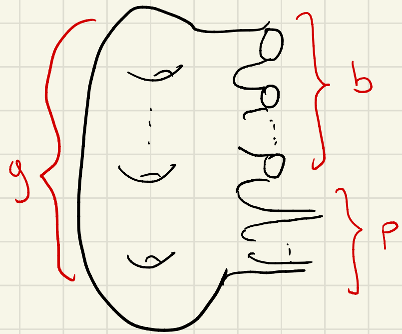
Superfici (iperboliche)

$S_{g,b,p}$ =



$= S_g \setminus \{b$ dischi aperti & p punti disgiunti$\}$

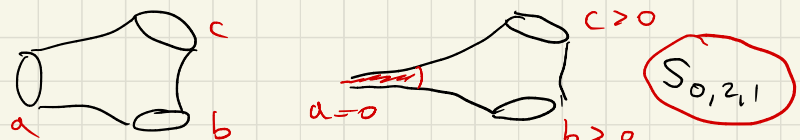
DI TIPO FINITO

$g,b,p \geq 0$

$\chi = 2-2g-b-p$

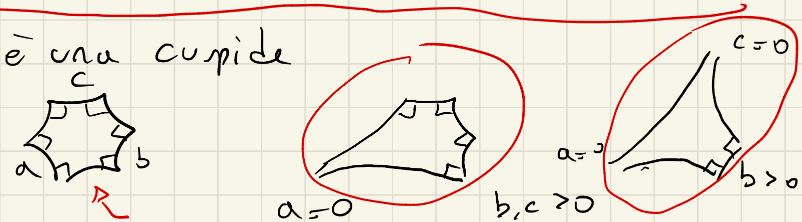
Prop: Se $\chi<0$, allora $S_{g,b,p}$ ammette metrica iperbolica completa di area finita e con ∂ geodetico.

Prop: $\forall a,b,c \geq 0$ $\exists!$

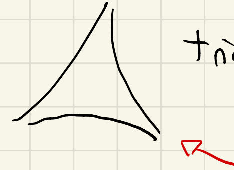


Si intende che se $a=0$ c'è una cuspide

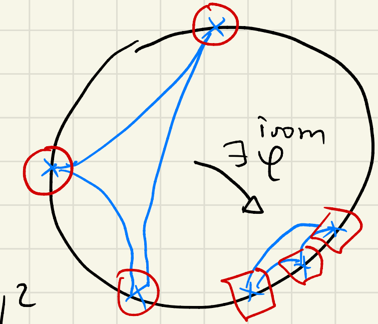
dim $\forall a,b,c \geq 0$ $\exists!$



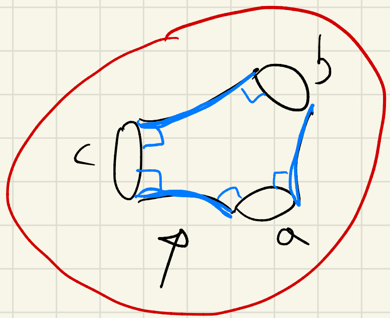
$a=b=c=0$

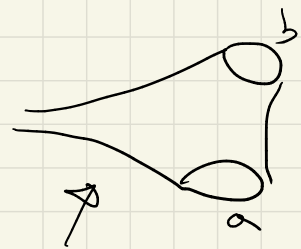


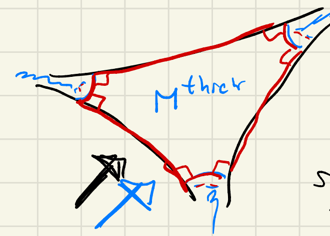
triangolo ideale $\exists!$



$\mathrm{Isom}^+(\mathbb{H}^2) = \mathbb{P}SL_2\mathbb{R}$ agisce transitiv.
sulle terne ordinate di punti in $\partial\mathbb{H}^2 = \mathbb{R}\cup\{\infty\}$

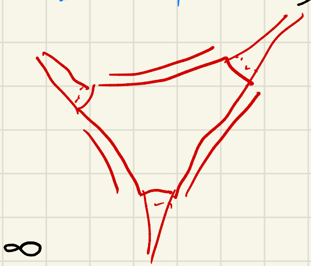






$S_{0,0,3}$

sfera con 3 punture



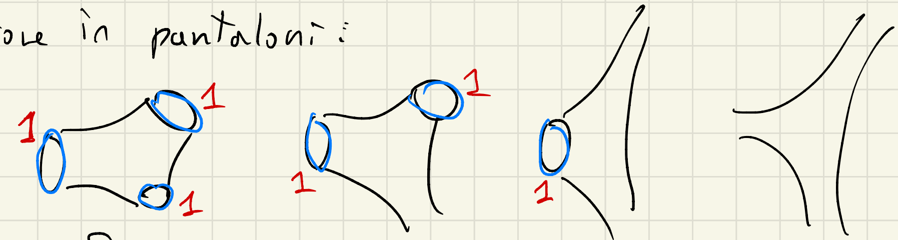
Prop: Se $\chi(S_{g,b,p})<0$ allora

$S_{g,b,p}$ ha metrica ip. compl. area $<\infty$
e con ∂ geod.

dim:

$\chi < 0 \Rightarrow$ Decomposizione in pantaloni:

$\Leftarrow \quad |\chi|$

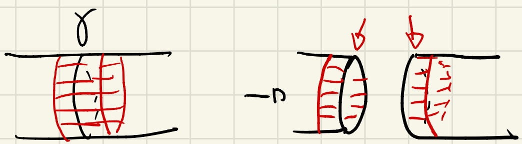


$S_{0,3,0}$ $\qquad 0,2,1 \qquad 0,1,2 \qquad 003$

$\chi = -1 \qquad -1 \qquad -1 \qquad -1 \qquad \square$

Ho 2 gradi di libertà per ogni curva della dec.

Def: Dec. in pants $:= \gamma_1, \dots, \gamma_k \subseteq S$ c.s.c. disgiunte tale che tagliando lungo $\gamma_1 \cup \dots \cup \gamma_k$ ottengo pantaloni



Ottengo $\overbrace{\qquad}^{2k+b}$ gradi di libertà

Teo: In questo modo si ottengono tutte le metriche iperboliche possibili su $S_{g,b,p}$.

CURVE SU SUPERFICI

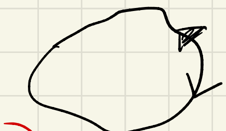
S superficie = 2-varietà

$S_{g,b,p}$

$\mathcal{S} = \mathcal{S}(S) = \{$ c.s.c. non banali non orientate a meno di isotopia $\}$

c.s.c. = curve semplici chiuse

NO

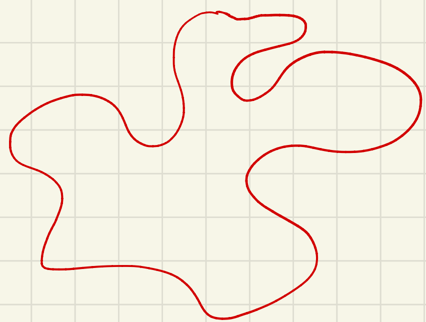


$\mathbb{R}^2 = S_{0,0,1}$

$D^2 = S_{0,1,0}$

Prop: $\mathcal{S}(S^2)$, $\mathcal{S}(\mathbb{R}^2)$, $\mathcal{S}(D^2)$ sono tutti vuoti (Jordan)

dim: $\mathbb{R}^2$



Taglio lungo C

$S \setminus C = S_1 \sqcup S_2$, $S_1 \to D^2$

H_1 banale $\Rightarrow$ C separa

Oss: S chiusa $\left[\; C \subseteq S$ sup. c.s.c. $\right.$

$\to$ C separa $:=$ $S \setminus C$ ha 2 c.c. $\iff [C] = 0$

$\to$ C non separa $:=$ $S \setminus C$ ha 1 c.c. $\iff [C] \neq 0$ in $H_1(S, \mathbb{Z})$

S_g

C non separa

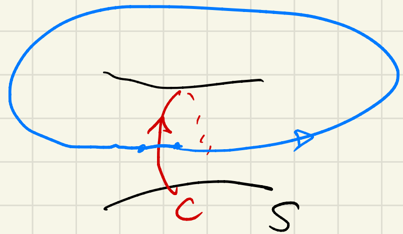
$\Updownarrow$

$\exists\, C' \subseteq S$

$C \pitchfork C' = \{pt\}$ $\implies$

$C \cdot C' = \pm 1$

$[C'], [C] \in H_1(S)$ non banali



$H_1(S) = \mathbb{Z}^{2g} + \dots$

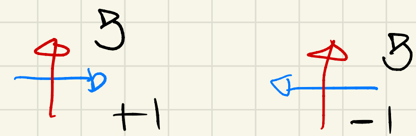
$H_1(S) \times H_1(S) \to \mathbb{Z}$
$\alpha \qquad \beta$

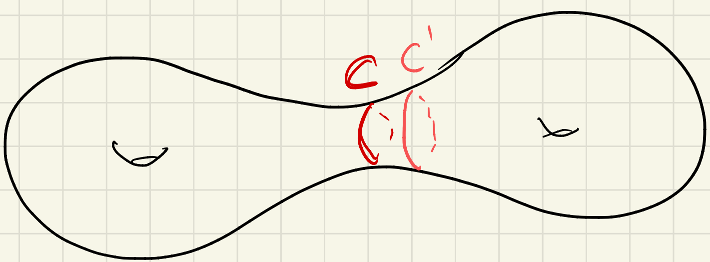
forma di intersezione

ANTISIMM.

$\alpha = [C_1] + \dots + [C_k]$

$\beta = [C_1'] + \dots + [C_h']$





$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$

$\implies [C] \cdot [C] = 0$

Prop: T toro $\mathcal{S}(T) \cong \mathbb{Q} \cup \{\infty\}$
$\quad\; \overset{\shortparallel}{S^1 \times S^1}$

Dato $(m,n) \in H_1(T) = \mathbb{Z} \times \mathbb{Z}$, $\exists!$ rappresentante che sia un curva s.c. se

a meno di isotopia

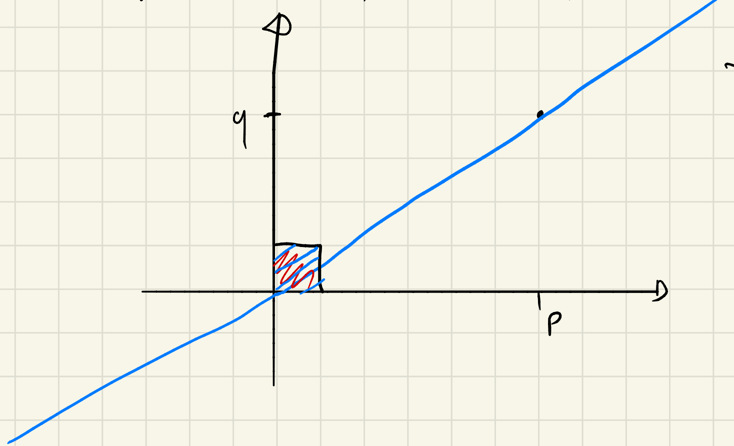
$MCD(m,n) = 1$, nessuna se $MCD(m,n) > 1$

oppure se $(0,0)$

dim:

- $(m,n) = (p,q) \in H_1(T) = \pi_1(T)$

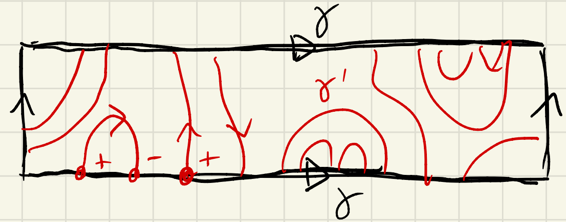
$\exists$:



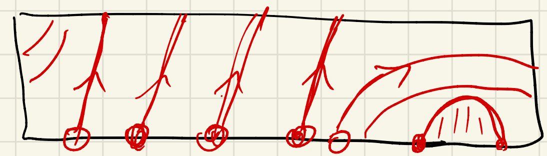
c.s.c. nel toro γ

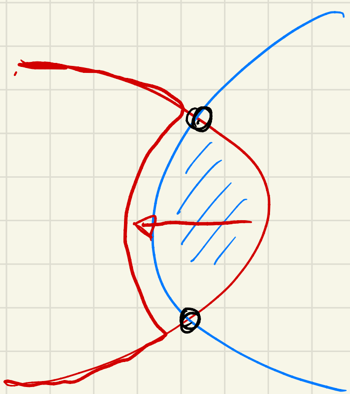
$!$: $[\gamma'] = [\gamma] = (p,q)$

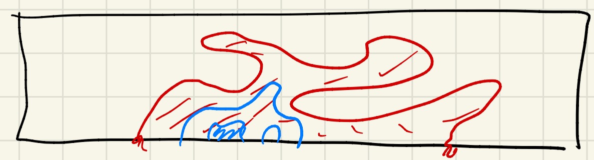
Taglio T lungo γ $\dashrightarrow$ anello $S^1 \times [0,1] = A$

$A =$ 

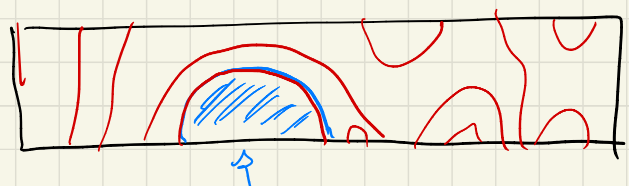
$\gamma' \pitchfork \gamma$

$\gamma \cdot \gamma' = 0 \implies$ 

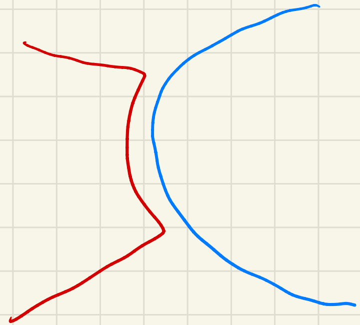




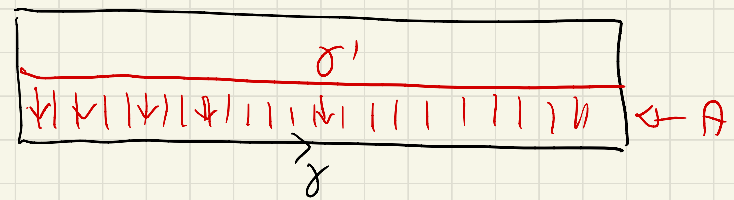
INNERMOST



BIGONO



$\#\{\gamma \cap \gamma'\}$ cala



(m,n) non coprimi $\Rightarrow$ no c.s.c.

$(0,0)$

Se C c.s.c. $\qquad [C] = (m,n) \neq (0,0)$

$\Rightarrow C$ non separa $\Rightarrow \exists\, C' \cdot C = \pm 1$

$\Rightarrow [C] \in H_1(S)$ è primitiva

$[C] = 3\alpha \qquad [C]\cdot[C'] = 3d$

$\square$

$H_1(T) = \mathbb{Z} \times \mathbb{Z}$

(p,q) c.s.c. $\rightarrow (-p,-q)$ $\rightarrow \mathcal{S} = \underbrace{\left\{\frac{p}{q}\right\}}_{\mathbb{Q}} \cup \{\infty\}$

Caso S_g $g \geq 2$ è più complesso $\wp$

<u>Prop:</u> In $S_{g,b,p}$ $\exists < +\infty$ c.s.c. a meno di DIFFEO di $S_{g,b,p}$

<u>Es:</u> In T $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $A \in SL(2, \mathbb{Z})$

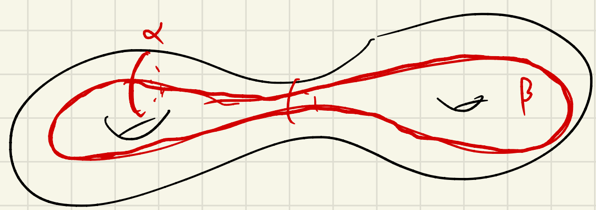
$$A : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \to T^2 \to T^2 \qquad T^2 = \mathbb{R}^2 / \mathbb{Z}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\mathbb{Z}^2 \xrightarrow{\sim} \mathbb{Z}^2$$

In omologia agisce proprio come A

$(p,q) = 1$

$$A = \begin{pmatrix} p & r \\ q & s \end{pmatrix} : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} p \\ q \end{pmatrix} \qquad H_1(T) = \mathbb{Z} \times \mathbb{Z} \xrightarrow{A} \mathbb{Z} \times \mathbb{Z} = H_1(T)$$

<u>Caso generale</u>

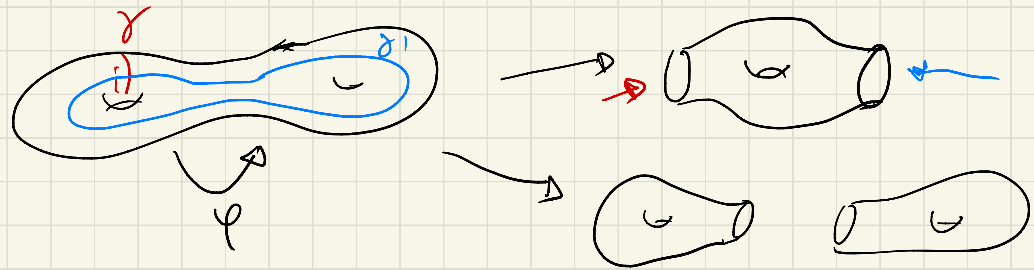


uso class. superfici !

$C, C' \subseteq S$ $\qquad$ $S/\!/_{C} \longrightarrow S/\!/_{C'}$ due superfici

$S_{g,b,p}$

c'è un numero finito di casi possibili



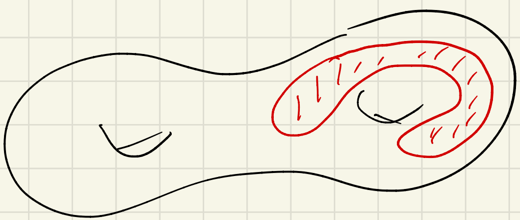
Prop: $S = S_g$

a) $\gamma \subseteq S$ c.s.c. omot. banale $\Leftrightarrow \gamma = \partial D^2$

b) " " non banale $\Rightarrow [\gamma] \in \pi_1(S, pt)$ è primitiva

c) " " " $\Rightarrow \gamma \not\sim \gamma^{-1}$

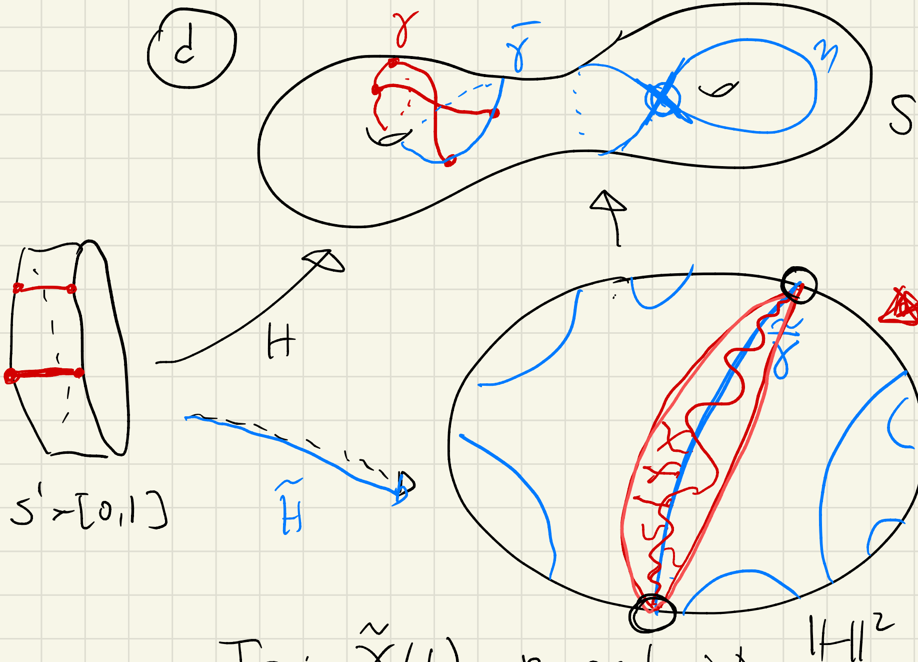
d) " " " Se S ha metrica iperbolica il suo rappresentante geod. è semplice



$[S^1, S]$

dim:

d



$\gamma \overset{H}{\dashrightarrow} \bar{\gamma}$

$[\gamma] \in \pi_1(S)^c$

$H : S^1 \times [0,1] \to S$

$H_0 = \gamma$

$H_1 = \bar{\gamma}$

<u>Ts</u>: $\tilde{\gamma}(t) \xrightarrow[t \to \pm\infty]{}$ endpoints di $\tilde{\gamma}$

Quindi $\bar{\gamma}$ è semplice: p. assurdo $\bar{\gamma}$ non semplice